# Plasma II (PHYS-424)

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Based on lecture notes by
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Swiss
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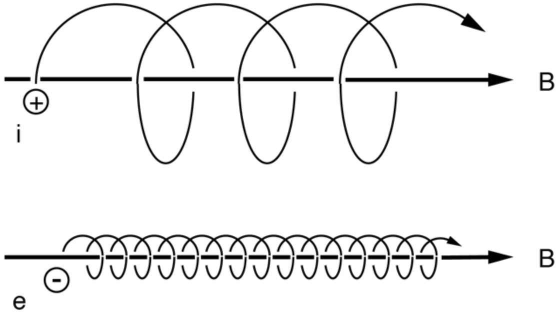
# Fusion energy - Recap

- **D-T fusion** is the most accessible fusion reaction for energy production
- Fusion fuel is in the state of a **plasma** with a temperature of at least 10keV (corresponding to 100 million degrees)
- **Confinement quality** $\boldsymbol{n\tau_E}$ must exceed $10^{20}\mathrm{m}^{-3}\mathrm{s}$ for fusion power to exceed power losses (break-even)

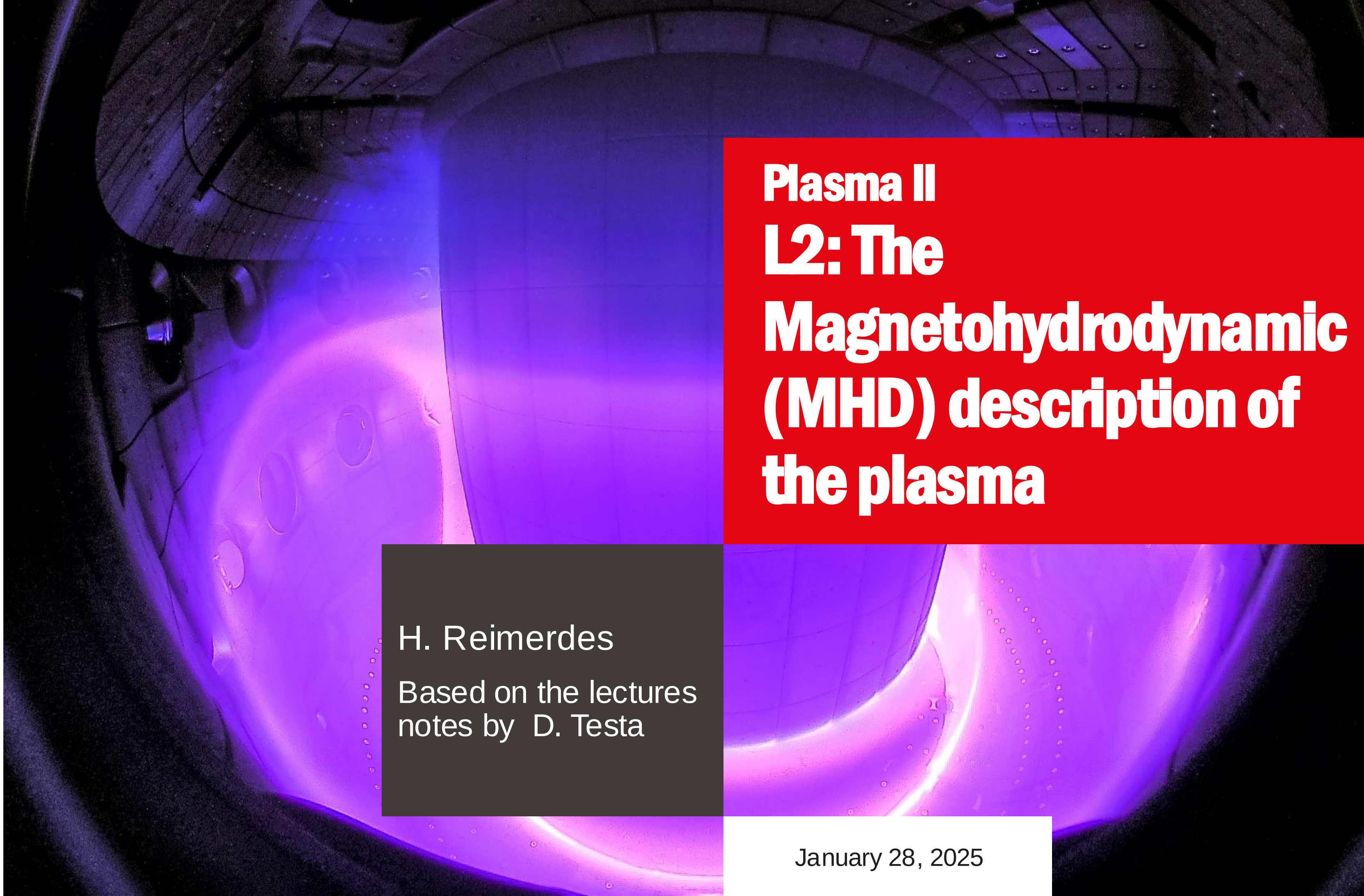
Use magnetic field to restrict movement of charged particles in the *perpendicular* direction and increase $\tau_E$ → **magnetic confinement fusion (MCF)**



# General introduction to the MHD lectures (L2 to L4)

- The Magnetohydrodynamic (MHD) description of a plasma
- MHD equilibrium configurations of interest for magnetic confinement fusion
- MHD stability and operational limits

- Go to https://moodle.epfl.ch/course/view.php?id=14996 for links to notes and exercises



# Plasma II

# L2: The Magnetohydrodynamic (MHD) description of the plasma

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Based on the lectures notes by  D. Testa

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# The Magnetohydrodynamic (MHD) description of the plasma

## Outline

- MHD equations
- General application
    - Validity
    - Conservation properties
- General formulation of the ideal MHD equilibrium
- The ideal MHD equilibrium in 1D configurations

## See also

- EPFL MOOC *Plasma physics: Introduction* #3a,e,f
    - https://learning.edx.org/course/course-v1:EPFLx+PlasmaIntroductionX+3T2016/home
- Wesson, *Tokamaks*, Chapters 2.18-2.21, 3.1-3.8
- Fitzpatrick, *Plasma Physics – An introduction*, Chapter 4

# Description of a plasma

- The physical foundation of MHD

  1. From single-particle description to distribution function
  2. The moment approach
  3. Single fluid description of the plasma

  \+ Maxwell’s equations

# From single particle description to distribution function

- **Start:** (microscopic) single-particle description with an equation of motion for each plasma particle under the influence of the Lorentz force

$$m\frac{d\bar{v}(\bar{x},t)}{dt} = \bar{F}_{\mathrm{L}}(\bar{x},t) = q[\bar{E}(\bar{x},t) + \bar{v}(\bar{x},t) \times \bar{B}(\bar{x},t)]$$

  - Exact description of the plasma!

- However, the plasma density is typically $10^{20}$m$^{-3}$ in magnetically confined fusion devices and the plasma volume is 100m$^3$

⬇

- $10^{22}$ particles, hence, $10^{22}$ individual equations of motion are simply computationally too expensive to solve!
  - Precise state cannot be measured in practice

# From single particle description to distribution function (cont.)

- Consider a *distribution function*

$$f_s(\bar{x}, \bar{v}, t)$$

which provides the number of particles of the plasma species $s$ occupying on average at the time $t$ the 6D *phase space volume*

$$dxdydz \times dv_x dv_y dv_z$$

centred at the 3D position $\bar{x}$ and the 3D velocity $\bar{v}$

  - The usual normalisation of $f_s$ is

Particle density: $$n_s(\bar{x}, t) = \int f_s(\bar{x}, \bar{v}, t) dv_x dv_y dv_z$$

and

Number of particles: $$N_s(t) = \int n_s(\bar{x}\,, t) dxdydz$$

# From single particle description to distribution function (cont.)

- Applying conservation of particles $\frac{df_s(\bar{x},\bar{v},t)}{dt} = 0$ (*Liouville's theorem*)

*Boltzmann/Vlasov kinetic description* of a plasma

- If $n\lambda_D^3 \gg 1$, local electric fields are shielded and $\bar{E}$ (and $\bar{B}$) correspond to macroscopic fields → collisionless *Vlasov equation*

# MHD: the **moment** approach

- The *kinetic description* of a system (e.g. fluid, plasma) $f_S(\bar{x}, \bar{v}, t)$ still does not have a simple meaning in terms of the usual macroscopic quantities

⬇

Apply mathematical operations to the kinetic Boltzmann/Vlasov equation to reduce the dimensionality and extract useful/intuitive/measurable quantities (fluid variables and equations)

# Moments of the distribution function yield fluid variables

- Velocity space moments of the distribution function $f_S(\bar{x},\bar{v},t)$

N$^{th}$ order moment: $$\frac{1}{\int_{-\infty}^{\infty} f_s(\bar{x},\bar{v},t)d\bar{v}} \int_{-\infty}^{\infty} \bar{V}^N\, f_s(\bar{x},\bar{v},t)\, d\bar{v} = \frac{1}{n_s(\bar{x},t)} \int_{-\infty}^{\infty} \bar{V}^N\, f_s(\bar{x},\bar{v},t)\, d\bar{v}$$

- *N*=0 moment ➔ particle density $n_s(\bar{x},t)$ (normalization of the distribution function $f_S(\bar{x},\bar{v},t)$ )
- *N*=1 moment: $\boldsymbol{\bar{v}}$ ➔ flow velocity $\bar{V}_s(\bar{x},t)$
- *N*=2 moment: $\boldsymbol{m_s\bar{v}\bar{v}}$ ➔ stress tensor $\bar{\bar{P}}_s(\bar{x},t)$ (pressure tensor, if calculated in the rest frame*)
- *N*=3 moment: $\boldsymbol{\frac{1}{2}m_s v^2\bar{v}}$ ➔ energy flux density $\bar{Q}_s(\bar{x},t)$ (heat flux density, if calculated in rest frame*)

*Replace $\boldsymbol{\bar{v}}$ with $\boldsymbol{\bar{v}} - \bar{V}_s$

# Moments of the kinetic equation yield fluid equations

- Kinetic “Boltzmann” equation

$$\frac{\partial f_s}{\partial t} + \bar{v} \cdot \nabla f_s + \bar{a} \cdot \nabla_v f_s = C_s(f)$$

$$\nabla \cdot (\bar{v} f_s) \qquad \nabla_v \cdot (\bar{a} f_s)$$

*N*=0 moment ➔ continuity equation

# Moments of the kinetic equation yield fluid equations

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$$\nabla\cdot(\bar{v} f_s) \qquad\qquad \nabla_v\cdot(\bar{a} f_s)$$

- *N*=0 moment ➔ continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\bar{V}_s) = 0$$

- *N*=1 moment ➔ momentum conservation equation

$$\frac{\partial(m_s n_s\bar{V}_s)}{\partial t} + \nabla\cdot\bar{\bar{P}}_s - q_s n_s(\bar{E} + \bar{V}_s\times B) = \bar{F}_s$$

- *N*=2 moment ➔ energy conservation equation

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_s + \frac{1}{2}m_s n_s V_s^2\right) + \nabla\cdot\bar{Q}_s - q_s n_s\bar{E}\cdot\bar{V}_s = W_s + \bar{V}_s\cdot\bar{F}_s$$

With $s = \mathrm{e},\mathrm{i}$
and $\bar{F}_\mathrm{e} = -\bar{F}_\mathrm{i}$

# Fluid closure

- Fluid equations require a “Fluid closure” ➔ assume only small deviations from a Maxwellian distribution

➢ Closures yield expressions for $\bar{F}$, $\bar{\bar{\pi}}_e$, $\bar{\bar{\pi}}_i$, $W_e$, $W_i$ , $\bar{q}_e$, $\bar{q}_i$ that depend on $n$, $T_e$ and $T_i$

- Distinguish $\dfrac{\rho_{e/i}}{\lambda_{\mathrm{MFP},e/i}} \begin{cases} \gg 1 : & \text{un-magnetised plasma} \\ \ll 1 : & \text{magnetised plasmas} \end{cases}$

  - Gyro radius $\rho_{e/i} \propto \dfrac{T_{e/i}^{1/2}}{B}$
  - Mean free path $\lambda_{\mathrm{MFP},e/i} \propto \dfrac{T_{e,i}^{2}}{n}$

$$\left.\begin{array}{l} \rho_{e/i} \propto \dfrac{T_{e/i}^{1/2}}{B} \\ \lambda_{\mathrm{MFP},e/i} \propto \dfrac{T_{e,i}^{2}}{n} \end{array}\right\} \quad \frac{\rho_{e/i}}{\lambda_{\mathrm{MFP},e/i}} \propto \frac{n}{BT_{e/i}^{3/2}}$$

# Order terms in equation by size and neglect "small" terms

- Simplify equations by neglecting terms that are small for a particular problem in physics, e.g. a collisional ($\lambda_{\mathrm{MFP,e/i}} \ll L$), highly magnetised ($\rho_{\mathrm{e/i}} \ll \lambda_{\mathrm{MFP,e/i}}$), quasi-neutral ($\lambda_{\mathrm{D}} \ll L$) plasma

$$\frac{\rho_e}{L}, \frac{\rho_i}{L} \ll \frac{\rho_e}{\lambda_{\mathrm{MFP,e}}}, \frac{\rho_i}{\lambda_{\mathrm{MFP,i}}}, \frac{m_e}{m_i} \ll 1$$

  - Characteristic length $L$

- MHD ordering: $V \sim v_{th}$
  - MHD motion is sufficiently fast that "transport" effects such as viscosity and thermal conductivity are too slow to play a role
  - The only collisional effects are resistivity, thermal force and electron-ion collisional energy exchange

# (2-fluid) MHD equations

- Electrons

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n\bar{V}_e) = 0$$

$$m_e n \frac{\partial \bar{V}_e}{\partial t} + m_e n(\bar{V}_e \cdot \nabla)\bar{V}_e + \nabla p_e + en(\bar{E} + \bar{V}_s \times B) = \bar{F}_U + \bar{F}_T$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \frac{3}{2}(\bar{V}_e \cdot \nabla)p_e + \frac{5}{2}p_e \nabla \cdot \bar{V}_e = S_{ie}$$

- Ions

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n\bar{V}_i) = 0$$

$$m_i n \frac{\partial \bar{V}_i}{\partial t} + m_i n(\bar{V}_i \cdot \nabla)\bar{V}_i + \nabla p_i - eZ_i n(\bar{E} + \bar{V}_i \times B) = -\bar{F}_U - \bar{F}_T$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \frac{3}{2}(\bar{V}_i \cdot \nabla)p_i + \frac{5}{2}p_i \nabla \cdot \bar{V}_i = -S_{ie}$$

# Linear combinations of ion and electron variables yield single fluid variables

- Define **single fluid variables** (linear combinations of ion "$i$" and electron "$e$" quantities) and use $n_i \approx n_e = n$ and $m_e/m_i \ll 1$
  - Mass density: $\rho_\mathrm{M} = nm_i + \cancel{nm_e} \approx nm_i$
  - Charge density: $\rho = e(n_i - n_e) \approx 0$
  - Center-of-mass velocity: $\bar{V} = \frac{1}{\rho_M}(nm_i\bar{V}_i + \cancel{nm_e\bar{V}_e}) \approx \bar{V}_i$
  - Current density: $\bar{J} = en(\bar{V}_i - \bar{V}_e) \approx en(\bar{V} - \bar{V}_e)$

# Linear combinations of 2-fluid equations yield the MHD equations

- Continuity equation

$$\frac{dn}{dt} + n\nabla \cdot \bar{V} = 0$$

- Equation of motion (with $p = p_i + p_e$)

$$m_i n \frac{d\bar{V}}{dt} + \nabla p - \bar{J} \times \bar{B} = 0$$

- Ohm’s law - perfect conductivity ➔ ideal MHD

$$\bar{E} + \bar{V} \times \bar{B} = \frac{1}{en}(\bar{J} \times \bar{B} - \nabla p_e) + \eta \bar{J} = \mathbf{0}$$

# Linear combinations of 2-fluid equations yield the MHD equations (cont.)

- Energy evolution equation → adiabatic equation of state

$$\frac{3}{2}\frac{dp}{dt}+\frac{5}{2}p\nabla\cdot\bar{V}=0 \quad\Longrightarrow\quad \frac{d}{dt}\left(\frac{p}{n^{5/3}}\right)=0$$

Adiabatic index $\gamma$

- Equipartition of energy ($T_i = T_e$)

$$S_{ie}=0$$

# Maxwell's equations in the MHD limit

- **Maxwell's equations** describe the electro-magnetic and electro-static part of the MHD model
  - Gauss's law: $$\nabla \cdot \bar{E} = 0$$
  - Gauss's law for magnetism: $$\nabla \cdot \bar{B} = 0$$
  - Faraday's law: $$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
  - Ampère's law: $$\nabla \times \bar{B} = \mu_0 \bar{J} + \xcancel{\frac{1}{c^2}\frac{\partial E}{\partial t}}$$

Neglect displacement currents ($V \ll c$)

# Introduction to the MHD model: Summary

**Essential elements of the MHD model of the plasma:**

- **Single fluid description ➔**
  - Considers the global plasma behaviour
  - Plasma inertia is provided by the ions and the fluid velocity is the net ion flow
  - Movement of electrons relative to the ions constitute a plasma current
- **Maxwell’s equations ➔**
  - Describe the electro-magnetic (EM) and electro-static (ES) components of the model
  - Link the sources of the EM field (charge and current densities) to the fields in the plasma

# The Magnetohydrodynamic (MHD) description of the plasma

## Outline

- MHD equations
- **General application**
  - Validity
  - Conservation properties
- General formulation of the ideal MHD equilibrium
- 1D configurations

# Validity of ideal MHD

- Key assumptions
  - Small Lamour radius

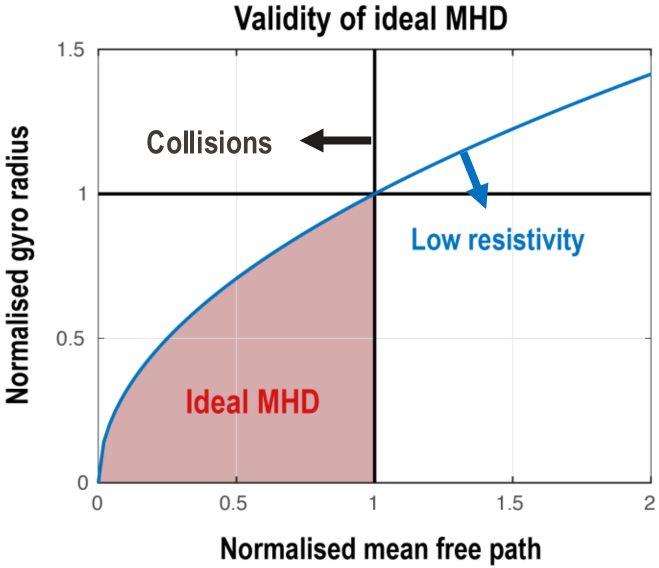
$$\frac{\rho_i}{L} \ll 1$$

  - Sufficient collisions

$$\sqrt{\frac{m_i}{m_e}}\frac{\lambda_i}{L} \ll 1$$

  - Sufficient low resistivity

$$\left(\frac{\rho_i}{L}\right)^2 \sqrt{\frac{m_e}{m_i}}\frac{L}{\lambda_i} \ll 1$$



# Conservation properties of the ideal MHD model

- The ideal MHD description of the plasma has many **conservation properties**

$$\frac{\partial(\text{something})}{\partial t} + \nabla\cdot(\text{flux of something}) = 0$$

- Conservation of mass (particles) $\rho$
- Conservation of momentum $\rho\bar{V}$
- Conservation of energy $\rho V^2/2 + B^2/(2\mu_0) + p/(\gamma-1)$
- Conservation of the magnetic flux $\Phi = \int_S \bar{B}\cdot d\bar{S}$

# Conservation of the magnetic flux

- Conservation of the magnetic flux $\Phi = \int_S \bar{B} \cdot d\bar{S}$ ➔ show $d\Phi/dt$=0

# Conservation of the magnetic flux

- Conservation of the magnetic flux $\Phi = \int_S \bar{B} \cdot d\bar{S}$ ➔ find when *d*Φ/*dt*=0!

$$\frac{d\Phi}{dt} = -\int_C \bar{E} \cdot d\bar{l} + \int_S \bar{B} \cdot \left(\bar{V} \times d\bar{l}\right) = -\int_C \underbrace{\left(\bar{E} + \bar{V} \times \bar{B}\right)}_{=\,0} \cdot d\bar{l}$$

Ideal Ohm’s law

- Magnetic flux through every surface moving with the plasma is constant

➔ Frozen-in magnetic field in the absence of plasma resistivity!

# Conclusions: when do we use the MHD model?

- The MHD model of the plasma can be used to:
  - Find magnetic field configurations capable of confining a plasma in a defined volume for sufficiently long time scales → equilibrium
  - Analyse the linear stability properties of such a magnetic equilibrium: find if the equilibrium is stable or unstable to (small) perturbations
  - Analyse the non-linear development of macroscopic and microscopic instabilities and their ultimate consequences on the equilibrium itself

# The Magnetohydrodynamic (MHD) description of the plasma

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- MHD equations
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    - Validity
    - Conservation properties
- **General formulation of the ideal MHD equilibrium**
- 1D configurations

# Definition of the (static) MHD equilibrium

- Finding stable equilibria is one of the essential applications of the ideal MHD model
  - Eulerian definition of the equilibrium: $\partial/\partial t = 0$
- The simplest, yet very useful, stable MHD equilibria are the static equilibria, i.e. $\bar{\boldsymbol{V}} = \mathbf{0}$
  - Dynamic MHD equilibria exist, but are more difficult to describe

# Static ideal MHD equilibrium equations

- Use ideal MHD equations and set $\partial/\partial t = 0$ and $\bar{V} = 0$
  - Continuity: $$\xcancel{\frac{\partial n(\bar{x},t)}{\partial t} + \nabla\cdot\left(n(\bar{x},t)\,\bar{V}(\bar{x},t)\right)} = 0$$
  - Momentum: $$m_i n\left(\xcancel{\frac{\partial\bar{V}(\bar{x},t)}{\partial t} + \bar{V}(\bar{x},t)\left(\nabla\bar{V}(\bar{x},t)\right)}\right) = \bar{j}(\bar{x},t)\times\bar{B}(\bar{x},t) - \nabla p(\bar{x},t)$$
  - Ampère's law: $$\nabla\times\bar{B}(\bar{x},t) = \mu_0\bar{j}(\bar{x},t)$$
  - Gauss's law: $$\nabla\cdot\bar{B}(\bar{x},t) = 0$$
  - Ideal Ohm's law: $$\bar{E}(\bar{x},t) + \xcancel{\bar{V}(\bar{x},t)\times\bar{B}(\bar{x},t)} = 0$$

# Static MHD equilibrium: isobaric surfaces

- Investigate momentum or force balance equation

$$\bar{j} \times \bar{B} = \nabla p$$

➢ Current and magnetic field lines lie on isobaric surfaces!

# Static MHD equilibrium: flux surfaces

Isobaric surface: surface of constant pressure $p(\bar{x},t)$

- $\bar{j}, \bar{B} \perp \nabla p$ ➔ $p$ does not change along $\bar{j}$ nor $\bar{B}$ ➔ $\bar{j}(\bar{x},t)$ and $\bar{B}(\bar{x},t)$ lie on isobaric surfaces

Flux surface

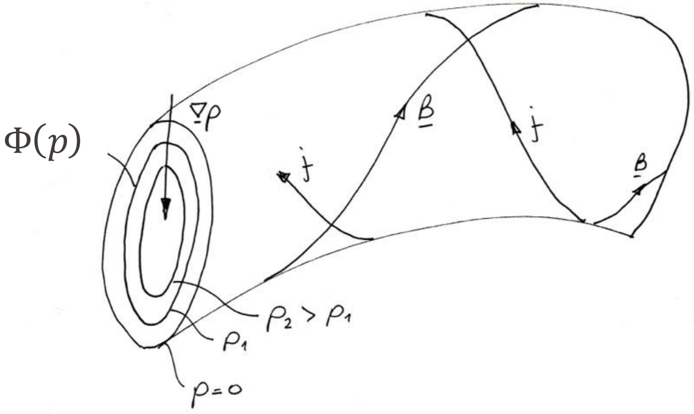
- $\nabla \bar{B} = 0$ implies that isobaric surfaces enclose constant magnetic flux $\Phi = f(p)$ ➔ isobaric flux-surfaces

# Definition of flux tubes

- Flux tube: a *cylindrical* volume embedded in the plasma, its sides are defined by magnetic field lines

- Frozen-in magnetic field in ideal MHD: $d\Phi/dt = 0$

  For any plasma displacement the integrity of the flux-tubes is maintained



- Ideal MHD: magnetic field topology is maintained
  - Field lines cannot break and “reconnect”

# Static MHD equilibrium

**MHD equilibrium**: $\bar{j} \times \bar{B}$ force balances the $\nabla p$-force

$$\bar{j} \times \bar{B} - \nabla p \;= 0$$

- Use Ampère's law

$$\frac{1}{\mu_0}(\nabla \times \bar{B}) \times \bar{B} - \nabla p = 0$$

$$\Rightarrow \nabla p + \nabla_\perp \left(\frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}(\hat{e}_B \cdot \nabla)\hat{e}_B = 0$$

magnetic field pressure

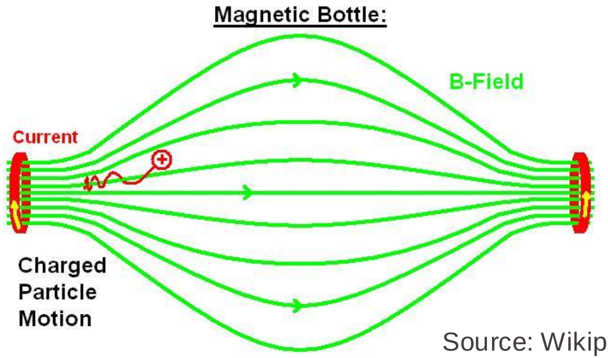
tension of magnetic field lines

- Balance between the gradient in the plasma + magnetic field pressure and the tension of the magnetic field lines
  - Tension of the magnetic field lines $\propto B^2/(\mu_0 R_C)$: restoring force when magnetic field lines are bent with a curvature radius $R_C$

# MHD equilibrium: field lines

- Two classes of MHD equilibria: field lines are open or closed inside the plasma volume being studied

## Examples



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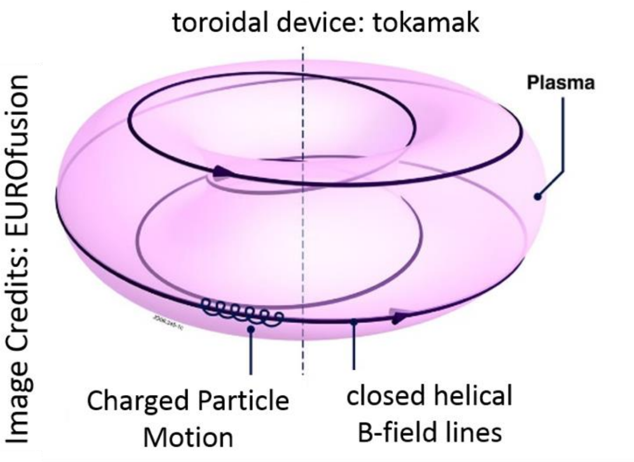

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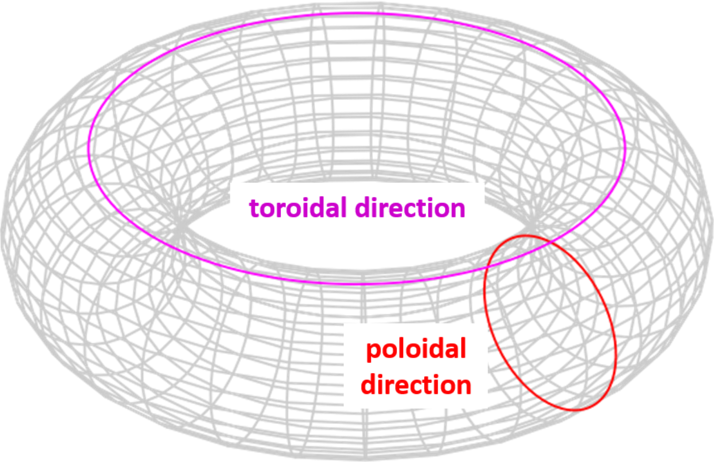
- Open magnetic field lines in a linear cylindrical device: the magnetic bottle (or magnetic mirror)

- Closed magnetic field lines in a toroidal device: the tokamak

# Closed field lines in a toroidal device

- A toroidal device (≡torus) is a cylinder with its ends folded-up
- Torus: two preferential directions defining possible symmetry axis:
  - The toroidal direction: the longitudinal axis of the *original* cylinder
  - The poloidal direction: defines the plane perpendicular to the longitudinal axis of the *original* cylinder



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- Magnetic field lines can be closed in a toroidal device

# MHD equilibrium: field lines & Poincaré theorem

- Poincaré theorem (also sometimes known as the *hairy ball theorem*): the only smooth 2D surface on which field lines can be covered by a non-vanishing vector field is a torus

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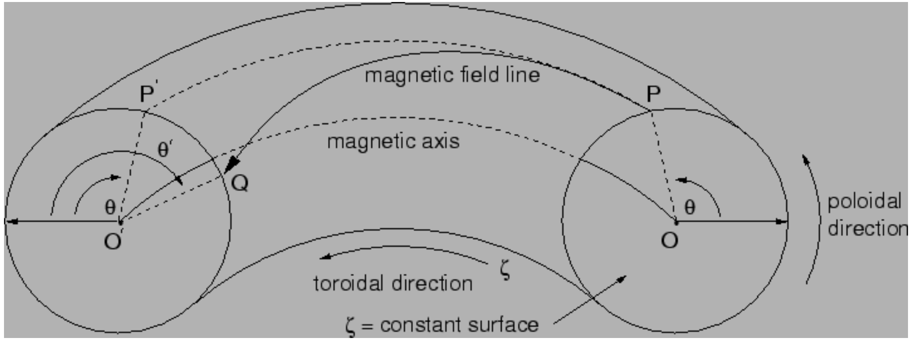
Vector field vanishes!

Flux surfaces must have the topology of a torus for the $\bar{j} \times \bar{B}$ force to balance the plasma pressure everywhere!

# Closed field lines: rotational transform

- Rotational transform ι: the average value of *d*θ after one full transit in the toroidal direction



$$\iota = \lim_{l \longrightarrow \infty} \frac{1}{l} \sum_{k=1}^{l} d\theta_k$$

$$= 2\pi \frac{n}{m} \qquad \{n, m\} \in \mathcal{N}$$

  - **Example:** if the field line closes itself after *m* toroidal transit and *n* of poloidal transits, then ι=2π*n*/*m*

➢ A perturbation can resonate with this natural periodicity of the magnetic equilibrium when $\iota = 2\pi z$ with $z \in \mathbb{Q}$

# Closed field lines: rotational transform

- Rotational transform ι(r): characterises flux surfaces of MHD equilibria with a single number and helps the study of MHD stability
  - Mostly used for non-axisymmetric 3D magnetic configurations ➔ stellarators

- Related to safety factor $q(r)=2\pi/\iota(r)$
  - Mostly used for axisymmetric 2D magnetic configurations ➔ tokamaks

# Static ideal MHD equilibrium: summary

- Static ideal MHD equilibrium: $\bar{V}(\bar{x},t)=0$
- The $\bar{j}\times\bar{B}$ force must balance the outward-directed $\nabla p$-force
- Isobaric flux-surfaces: surfaces of constant pressure and magnetic flux, field lines tangent to them
- Closed field lines: characterized by the rotational transform

# The Magnetohydrodynamic (MHD) description of the plasma

## Outline

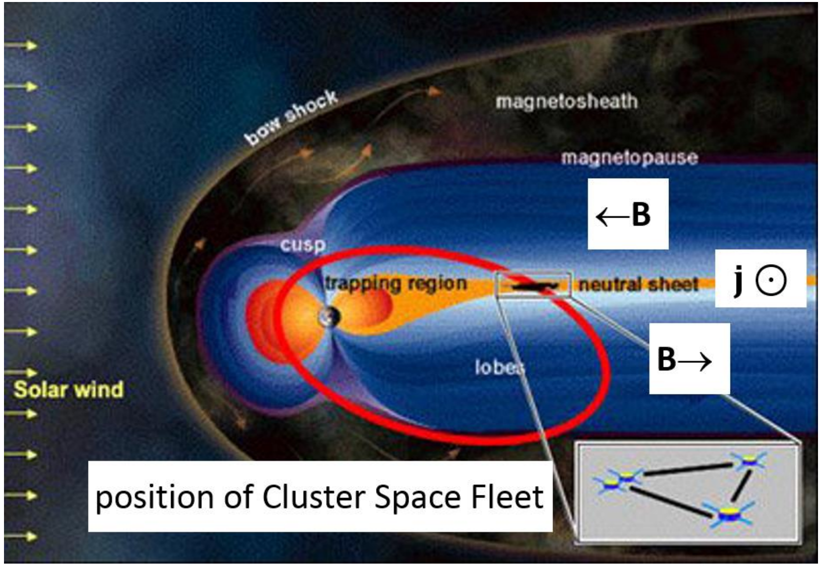
- MHD equations
- General application
  - Validity
  - Conservation properties
- General formulation of the ideal MHD equilibrium
- **The ideal MHD equilibrium in 1D configurations**
  - The Harris neutral sheet in the Earth’s magneto-tail
  - One of the earliest fusion experiments: the Z-pinch
  - One of the earliest fusion experiments: the $\theta$-pinch

# The Harris neutral sheet: a 1D MHD equilibrium

- **Harris neutral sheet**: on the night-side of the Earth’s magneto-sphere
  - Probed in recent years by the ESA Cluster and NASA Themis missions

  - Cross-field flow of current due to the solar wind: the direction of the magnetic field reverses as the mid-plane is crossed

  - The Harris neutral sheet corresponds to the reconnection layer, where the magnetic field reversal occurs



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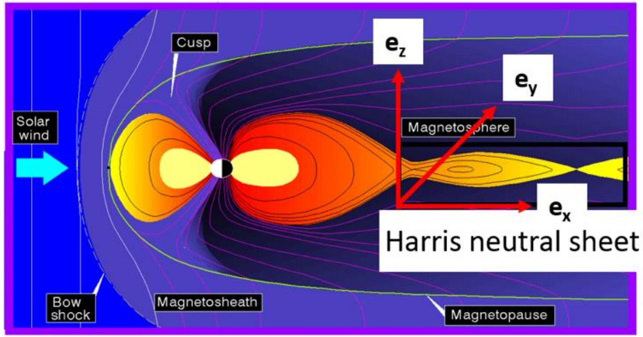
# The Harris neutral sheet: a 1D MHD equilibrium

- **Harris neutral sheet**
  - Symmetries: $\bar{B} = (B_x(z), 0, 0)$

    $\bar{\jmath} = \left(0, j_y(z), 0\right)$

➔ $\partial/\partial x = 0$ and $\partial/\partial y = 0$



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Ampère

$$\nabla \times \bar{B} = \mu_0 \bar{\jmath} \;\rightarrow\; \frac{dB_x(z)}{dz} = \mu_0\, j_y(z)$$

Force balance

$$\bar{\jmath} \times \bar{B} - \nabla p = 0 \;\rightarrow\; -B_x(z) j_y(z) - \frac{dp(z)}{dz} = 0$$

$$\left.\begin{aligned}&\frac{dB_x(z)}{dz} = \mu_0\, j_y(z)\\ &-B_x(z) j_y(z) - \frac{dp(z)}{dz} = 0\end{aligned}\right\} \quad \frac{d}{dz}\left(\frac{B_x^2(z)}{2\mu_0} + p(z)\right) = 0$$

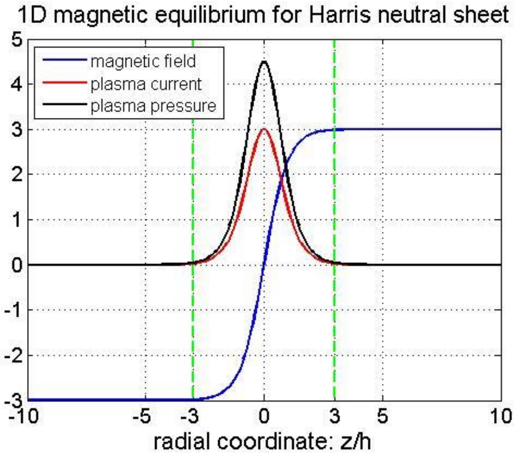
- The sum of the magnetic and the plasma pressure is constant in the Harris neutral sheet

# The Harris neutral sheet: a 1D MHD equilibrium

- General solution for the equilibrium of the Harris neutral sheet

$$B_x(z) = B_0 \tanh(z/h) \quad \left\{ \begin{array}{l} p(z) = \dfrac{B_0^2}{2\mu_0} \operatorname{sech}^2\left(\dfrac{z}{h}\right) \\[2ex] j_y(z) = j_0 \operatorname{sech}^2\left(\dfrac{z}{h}\right) \end{array} \right.$$

  - Derived by E.G. Harris in 1962
  - Confirmed by measurements in the Earth's magneto-tail



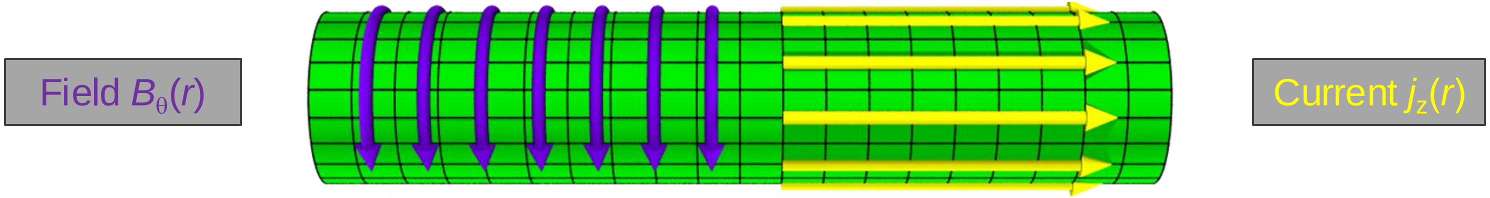
# The magnetic pinch: a 1D MHD equilibrium

- **Magnetic pinch**: 1D cylindrical (linear) configuration
- All equilibrium quantities depend only on the radial coordinate *r* ➔ use cylindrical coordinates ($e_r$, $e_\theta$, $e_z$) with $\partial/\partial z=0$ and $\partial/\partial\theta=0$
- Three varieties of magnetic pinches exist:
  - the **Z-pinch**
  - the **θ-pinch**
  - the **screw-pinch**

# The Z-pinch: a 1D MHD equilibrium

- **Z-pinch**: configuration that achieves plasma confinement using a poloidal magnetic field $B_\theta(r)$ and a 'toroidal' current $j_z(r)$

$$\begin{cases} \bar{B} = (0, B_\theta(r), 0) \\ \bar{J} = \left(0, 0, j_z(r)\right) \end{cases}$$



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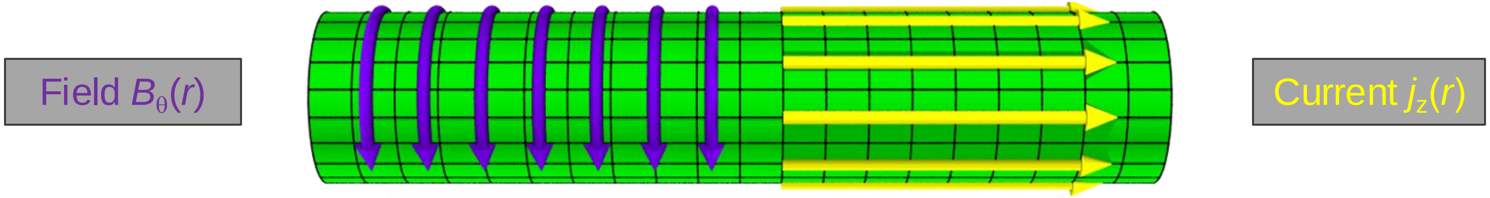
➢ See exercise 1 for Z-pinch equilibrium

# The Z-pinch: a 1D MHD equilibrium



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- Z-pinch equilibrium

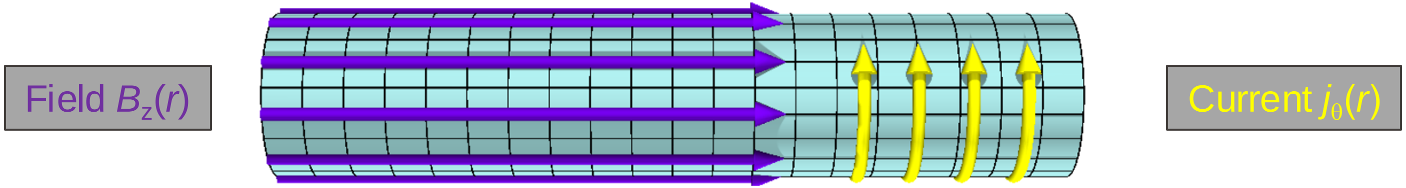
$$\frac{d}{dr}\left[p(r) + \frac{B_\theta^2(r)}{2\mu_0}\right] + \frac{B_\theta^2(r)}{\mu_0 r} = 0$$

  - The outward net force provided by the radial gradient of the plasma kinetic + magnetic pressure is balanced by the inward tension due to the curvature of the magnetic field lines being wrapped over the surface of the Z-pinch
    - Tension of the magnetic field lines $\propto B^2/(\mu_0 r)$: restoring force when magnetic field lines are bent with a curvature radius $r$

# The Θ-pinch: a 1D MHD equilibrium

- **θ-pinch**: configuration that achieves plasma confinement using a toroidal magnetic field $B_z(r)$ and a poloidal current $j_\theta(r)$

$$\begin{cases} \bar{B} = \big(0,0,B_z(r)\big) \\ \bar{J} = \big(0,j_\theta(r),0\big) \end{cases}$$



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Ampere's law: $\mu_0\bar{J} = \nabla\times\bar{B} \quad\Longrightarrow\quad \mu_0 j_\theta(r) = -\frac{dB_z(r)}{dr}$

Ideal MHD force balance $\bar{J}\times\bar{B} - \nabla p = 0 \quad\Longrightarrow\quad j_\theta(r)B_z(r) - \frac{dp(r)}{dr} = 0$

Substitute $j_\theta$ $\quad\Longrightarrow\quad \frac{1}{\mu_0}\frac{dB_z(r)}{dr}B_z(r) + \frac{dp(r)}{dr} = 0$

$$\Leftrightarrow\quad \frac{d}{dr}\left(\frac{B_z^2(r)}{2\mu_0} + p(r)\right) = 0 \quad\Longrightarrow\quad \boxed{\frac{B_z^2(r)}{2\mu_0} + p(r) = const. = \frac{B_z^2(a)}{2\mu_0}}$$

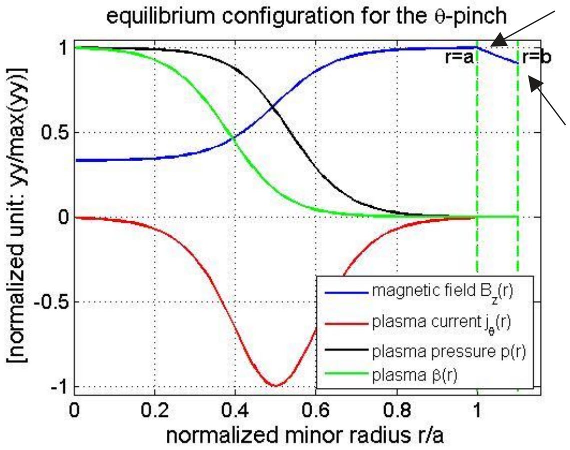
# The θ-pinch: a 1D MHD equilibrium

- **θ-pinch equilibrium**
  - Plasma kinetic pressure is balanced by magnetic field pressure
  - Analytic solution obtained by prescribing the toroidal field

$$\frac{B_z^2(r)}{2\mu_0} + p(r) = \frac{B_z^2(a)}{2\mu_0}$$

$$B_z(r) = \begin{cases} B_0\left[2 + \tanh\left(\frac{2r-a}{h}\right)\right] & \text{for } \; r < a \\ B_z(r=a)\,\frac{a}{r} & \text{for } \; a < r < b \end{cases}$$

with $p(r) = \frac{B_z^2(a) - B_z^2(r)}{2\mu_0}$



radius of the plasma column

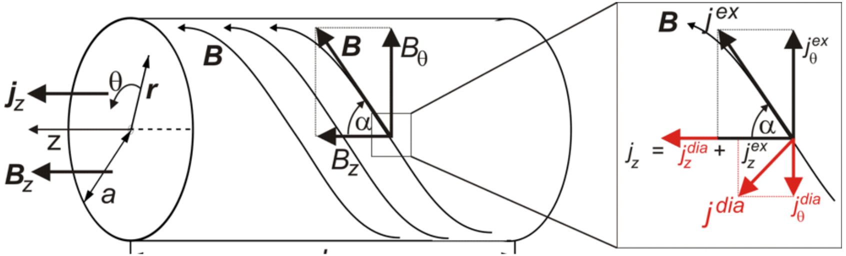
position of the limiter wall

$$p(r) = \frac{B_0^2(a)}{2\mu_0}\left\{\left[2 + \tanh\left(\frac{a}{h}\right)\right]^2 - \left[2 + \tanh\left(\frac{2r-a}{h}\right)\right]^2\right\}$$ and $$j_\theta(r) = -\frac{B_0}{h\mu_0}\,\mathrm{sech}^2\left(\frac{2r-a}{h}\right)$$

# The screw-pinch: a 1D MHD equilibrium

- **Screw-pinch**: combines the Z-pinch and the θ-pinch, also known as the linear tokamak

$$\begin{cases} \bar{B} = \left(0, B_\theta(r), B_z(r)\right) \\ \bar{j} = \left(0, j_\theta(r), j_z(r)\right) \end{cases}$$



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➢ See exercise 2 for screw-pinch equilibrium

$$\frac{d}{dr}\left(p + \frac{B_z^2 + B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

# Applications of the 1D MHD equilibrium: summary

- **Harris sheet** $\{j_y(z),\, B_x(z)\}$ ➔ magnetic pressure

$$\frac{d}{dz}\left(p(z)+\frac{B_x^2(z)}{2\mu_0}\right)=0$$

- **Z-pinch** $\{j_z(r),\, B_\theta(r)\}$, ➔ magnetic pressure + magnetic field lines tension

$$\frac{d}{dr}\left(p(r)+\frac{B_\theta^2(r)}{2\mu_0}\right)+\frac{B_\theta^2(r)}{\mu_0 r}=0$$

- **θ-pinch** $\{j_\theta(r),\, B_z(r)\}$ ➔ magnetic pressure

$$\frac{d}{dr}\left(p(r)+\frac{B_z^2(r)}{2\mu_0}\right)=0$$

- **Screw-pinch** $\{j_\theta(r),\, j_z(r),\, B_\theta(r),\, B_z(r)\}$ ➔ *combines Z-pinch and θ-pinch*

$$\frac{d}{dr}\left(p(r)+\frac{B_z^2(r)+B_\theta^2(r)}{2\mu_0}\right)+\frac{B_\theta^2(r)}{\mu_0 r}=0$$

# Re-cap: The Magnetohydrodynamic (MHD) description of the plasma

- MHD equations: concept, derivation, (limits of validity), applications

- General MHD equilibrium:
  - Based on ideal MHD force balance (Ampere's law links $\bar{\jmath}$ and $\bar{B}$)
  - Field lines wind around a torus
  - Field lines form flux surfaces characterised by a rotational transform

- MHD equilibrium: several examples of one-dimensional equilibria (require two symmetries, e.g. magnetic pinches with $\partial/\partial z=0$ and $\partial/\partial\theta=0$)